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# TWO-SIDED $L_p$ ESTIMATES OF CONVOLUTION TRANSFORMS

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**Let $f$ and $g$ be two Lebesgue measurable functions on the real line. Then the equation**

$$(f*g)(x) = \int_{-\infty}^{+\infty} f(t)g(x-t)dt$$

**defines the convolution transform of $f$ and $g$. In an earlier paper [4] we obtained sharp upper and lower estimates for the expression**

$$\text{(A)} \qquad \sup_{\substack{|E|\leq u \\ f_i \sim g_i^*}} \int_E |(f_1 * \cdots * f_n)(x)|^p d(x)$$

**where $p = 1, 2$ and 4, with applications to Fourier transform inequalities. This paper contains estimates of $(A)$ for all values of $p (p \geq 1)$ in the case where $E = (-\infty, +\infty)$. For example, one of our theorems implies the following:**

**"If $g_i^*$ is bounded and has compact support for all $i$, then there exists a constant $K$, $1/(p+1)^p \leq K \leq (p'+1)^p (2^{n-1})^p$, such that**

$$(A) = K \int_0^\infty |x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)|^p d(x) \text{ ."}$$

**Here $g_i^*$ are preassigned decreasing functions and the symbol $f_i \sim g_i^*$ means**

$$|\{x: |f_i(x)| > y\}| = |\{x: g_i^*(x) > y\}| \qquad \textbf{for all } y > 0 \text{ .}$$

Introduction. In an earlier paper [4], we obtained sharp upper and lower estimates for the expression

$$\text{(A)} \qquad \sup_{\substack{f_i \sim g_i^* \\ |E| \leq u}} \int_E \psi(f_1 * f_2 * \cdots * f_n)(x) d(x)$$

where $\psi(u) = u, u^2$. R. O'Neil obtained sharp upper and lower estimates when $\psi(u) = u$ and $n = 2$, [3, Lemma 1.5]. Our results coincide with his for this case.

We were able to apply our estimates for the case $\psi(u) = u^2$ ($n$-arbitrary) to classical Fourier transform inequalities of Hardy and Littlewood.

The main problem of this paper is to determine whether or not one can obtain the same types of upper and lower estimates for the functions $\psi(u) = u^p$, $p > 1$. We have, in fact, obtained such estimates for a class of functions $\psi$ containing the class $\psi(u) = u^p$. We are able to prove [see Theorem 2.12 and Corollary 2.18 of this paper] that there exist $p, q > 1$ such that

$$\begin{aligned}
&\frac{1}{(p'+1)^p}\cdot\frac{1}{(2^{n-1})^p}\int_0^\infty \psi[x^{n-1}(g_1^{**}-g_1^*)\cdots(g_n^{**}-g_n^*)(x)]d(x)\\
&\leqq \sup_{f_i\sim g_i^*}\int_{-\infty}^{+\infty}\psi((f_1*\cdots*f_n)(x)d(x)\\
&\leqq (q+1)^q\int_0^\infty \psi[x^{n-1}(g_1^{**}-g_1^*)\cdots(g_n^{**}-g_n^*)(x)]d(x)\ .
\end{aligned}$$

In proving our estimates, we use as a major tool Lemma 2.4 which contains the following inequality:

$$\left(\frac{1}{p+1}\right)\|g\|_p \leqq \left\|\frac{1}{x}\int_0^x g - g(x)\right\|_p \leqq (p'+1)\|g\|_p\ .$$

All the functions $f, g, \cdots$ which appear in this paper will be nonnegative, Lebesgue measurable functions for which $|\{x: f(x) > y\}| < \infty$ for every $y > 0$. By the statement $f(x) \doteq g(x)$ we mean that $|\{x: f(x) \neq g(x)\}| = 0$.

**I. Preliminaries.** The idea of considering the decreasing rearrangement $f^*$ and symmetrically decreasing rearrangement $\bar{f}$ of a function $f$ for finding sharp inequalities of convolution transforms was first noticed by Hardy, Littlewood and Pólya [1, Chapter X]. Since all our estimates are in terms of $f^*$ and $\bar{f}$, we shall start by defining these concepts.

DEFINITION 1.1 We say the functions $f$ and $g$ are equimeasurable; we write $f \sim g$, if

$$|\{x: f(x) > y\}| = |\{x: g(x) > y\}| \quad \text{for all} \quad y > 0\ .$$

DEFINITION 1.2 By $f^*(x)$, we denote a function such that

(i) $\qquad f^*(x)$ decreases for $x > 0$

and

(ii) $\qquad f^* \sim f\ .$

Further, for $x > 0$ we set,

$$f^{**}(x) = \frac{1}{x}\int_0^x f^*(t)dt\ ,$$

and finally, we set $\bar{f}(x) = f^*(2|x|)$.

In a similar manner, we can discuss the decreasing rearrangement of a sequence of nonnegative numbers $a_1, a_2, \cdots, a_n$. That is, we rearrange this sequence into a decreasing sequence. The new sequence is denoted by $a_1^*, a_2^*, \cdots, a_n^*$. This sequence is characterized by the following two properties:

$$\text{(i)} \qquad a_1^* \geqq a_2^* \geqq \cdots \geqq a_n^* ,$$

and

$$\text{(ii)} \qquad N(\{a_i \mid a_i > y\}) = N(\{a_i^* \mid a_i^* > y\}) \quad \text{for all} \quad y > 0 .$$

For a given set $A$, $N(A)$ stands for the number of points in $A$. Therefore, if $\psi \in C(-\infty, +\infty)$, $\psi$ is increasing, and $\psi(0) = 0$, we have

$$(1.3).^1 \qquad \sum_{k=1}^{n} \psi(a_i) = \sum_{k=1}^{n} \psi(a_i^*) .$$

Also, if $f(\geqq 0)$ is a step function with compact support, then $\int_{-\infty}^{+\infty} \psi(f(x))d(x) = -\int_0^{+\infty} \psi(y)d(m(f, y))$. Where the second term is a Riemann-Stieltjes integral with $m(f, y) = |\{x: f(x) > y\}|$. Now by a limiting argument, we see that

$$(1.4).^2 \qquad \int_{-\infty}^{+\infty} \psi(f(x))dx = \int_0^{\infty} \psi(f^*(x))d(x)$$

for all functions $f$ such that $m(f, y) < \infty$ for $y > 0$.

A nonnegative sequence $\langle \bar{a}_i \rangle_{-\infty}^{*\infty}$ is said to be symmetrically decreasing if $\bar{a}_0 \geqq \bar{a}_1 = \bar{a}_{-1} \geqq \cdots \geqq \bar{a}_n = \bar{a}_{-n} \geqq \cdots$. It is well-known [**1**, Theorom 375, p. 273] that the convolution of symmetrically decreasing sequences is again a symmetrically decreasing sequence. The previous statement also holds if the term "sequence" is replaced with the term "function".

LEMMA 1.5.[3] *The function $h_n(x) = \bar{g}_1 * \bar{g}_2 * \cdots * \bar{g}_n(x)$ is a symmetrically decreasing function.*

Let $(\bar{a}_i)$, $(\bar{b}_i)$ and $(\bar{c}_i)$ be given symmetrically decreasing sequences. If a nonnegative sequence $(a_i)$ can be rearranged to equal $(\bar{a}_i)$ term-for-term, then we write $(a_i) \sim (\bar{a}_i)$.

Pólya [**2**] had asked: When is the sum $S = \sum_{r+s+t=0} a_r b_s c_t$, for all rearrangements of the $a_r's$, $b_s's$, $c_t's$ (where $(a_i) \sim (\bar{a}_i)$, $(b_i) \sim (\bar{b}_i)$, $(c_i) \sim (\bar{c}_i)$) the greatest? Hardy and Littlewood answered this question [**2**] by proving the general statement

[1] This equation holds without any restriction on $\psi$; however, we need these restrictions in order for equation (1.4) to hold.

[2] This is the counterpart to (1.3) for functions.

[3] The proof can be found in [**4**].

$$\sum_{r+s+t+\cdots=0} a_r b_s c_t \cdots \leqq \sum_{r+s+t+\cdots=0} \bar{a}_r \bar{b}_s \bar{c}_t \cdots .$$

They were later able to prove the theorem for functions [1, Theorem 379; p. 279] which we now state.

THEOREM A (Hardy and Littlewood).[4]

$$\sup_{\substack{r_1 \sim r^* \\ f_i \sim g_i^*}} \int_{-\infty}^{+\infty} r_1(t)(f_1 * \cdots * f_n)(t)dt = \int_{-\infty}^{+\infty} \bar{r}(t)h_n(t)dt . \tag{1.6}$$

In (1.6) if we set $r^*(x) = \chi_{[0,u]}^{(x)}$, then the estimate on the right reduces to $\int_{-u/2}^{u/2} h_n(x)d(x)$. In Theorem B, we give our estimate of (1.6) which has many advantages over the above estimate.

THEOREM B.[3] *If $g_i^{**}$ is finite for each $x$, then there exists a constant $K$, $1/2^{n-1} \leqq K \leqq 1$, such that*

$$\begin{aligned} &\sup_{\substack{|E| \leqq u \\ f_i \sim g_i^*}} \int_E (f_1 * f_2 * \cdots * f_n)(x)d(x) \\ &= Ku \int_u^\infty x^{n-2}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)d(x) . \end{aligned} \tag{1.7}$$

Here, $K$ depends on $u$ and $g_1^*, g_2^* \cdots, g_n^*$.

If we set $R_n(x) = d/(d(x))\{x \int_x^\infty dv(v^{n-2}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*))$, then by combining Theorem A with Theorem B we see that there exists a constant $K$ $(1/2^{n-1} \leqq K \leqq 1)$ such that

$$\int_{-u/2}^{u/2} h_n(x)d(x) = K \int_0^u d(x)R_n(x) . \tag{1.8}$$ [5]

In general, the right side of (1.8) (our estimate) is easier than the left side to determine. For example, take $g_i^*(x) = 1/x^{\lambda_i}$, with $0 < \lambda_i < 1$. Thus, we see that $R_n(x)$ plays an important role in the solution of (1.7). In the next lemma, we state some important properties of this function.

LEMMA 1.9.[6] *If $t^{n-2}$ $(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*) \in L(a, \infty)$ for each $a > 0$, then the function $R_n(x)$ has the following properties*:

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[4] The case $n = 2$ appears in the cited reference; however, the general case is easily derivable from it.

[5] One of the properties of $R_n(x)$ is that

$$\int_0^u R_n(x)d(x) = u \int_u^\infty (x^{n-2}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)d(x) .$$

[6] This lemma can easily be verified.

(a) $R_n(x) \geqq 0, x \geqq 0,$
(b) $R_n(x)$ *decreases*, $x \geqq 0.$
(c) $\int_0^x R_n(t)dt = x\int_x^\infty dt(t^{n-2}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*)).$
(d) $\int_0^x R_n(t)dt - xR_n(x) \doteq x^n(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*).$

**II. Two-sided estimates.** As has been stated earlier, our main problem here is to show that there is a constant $K > 0$ such that

$$\sup_{f_i \sim g_i^*} \int_{-\infty}^{+\infty} \psi((f_1 * \cdots *)(x)d(x)$$
$$= K \cdot \int_0^\infty \psi(x^{n-1}(g_1^{**} - g_1^*) \cdots (g_n^{**} - g_n^*))d(x)$$

where $g_1^*, \cdots, g_n^*$ are preassingned decreasing functions and the supremum is taken over all $f_i$'s such that $f_i \sim g_i^*$. We obtain this result for $\psi$'s that are a proper subclass of the convex functions (see Definition 2.7). Let us begin by developing properties of this class of $\psi$'s.

DEFINITION 2.1. We say that a function $\psi \in \mathsf{V}(\Lambda)$ on $[a, b]$, if

1. $\psi(a) = 0$
2. $\psi'(t)$ exists for $t \in [a, b]$
3. $0 \leqq \psi'(t)$ and increases (decreases) for $t \in [a, b]$.

LEMMA 2.2. *If*

1. $\psi \in \mathsf{V}$ *on* $[0, \infty)$
2. $0 \leqq f, g \in L_1[0, u]$ *for each* $u > 0$
3. $\int_0^u f(t)dt \leqq \int_0^u g(t)dt$ *for* $u > 0$
4. $f(t)$ *decreases for* $t \geqq 0$, *then*
$\int_0^u \psi(f(t))dt \leqq \int_0^u \psi(g(t))dt$ *for* $u \geqq 0$.

*Proof.* First let us assume that $f$ and $g$ are continuous on $[0, \infty)$. Since $\psi \in \mathsf{V}$ we get by the Mean-Value-Theorem

$$\psi(g(t)) - \psi(f(t)) \geqq \psi'(f(t))(g(t) - f(t)) \,.$$

Therefore,

$$\int_0^u \psi(g(t)) - \psi(f(t))dt \geqq \int_0^u \psi'(f(t))(g(t) - f(t))dt$$

and by the second Mean-Value-Theorem for integrals there is a $0 \leqq \xi \leqq u$ such that,

$$= \psi'(f(0)) \int_0^\xi [g(t) - f(t)]dt \geqq 0 .$$

Now we apply a limiting argument to get the general case.

LEMMA 2.3. *If $\psi \in \mathbf{V}$ on $[0, \infty)$, then*

$$\sup_{f_i \sim g_i^*} \int_{-\infty}^{+\infty} \psi(f_1 * \cdots * f_n)(x)d(x) = \int_{-\infty}^{+\infty} \psi(h_n(x)d(x)) .$$

*Proof.* In Theorem A set $r^*(t) = \chi_{[0,u]}(t)$, then we find that

$$\int_{-\infty}^{+\infty} \chi_E(x)(f_1 * \cdots * f_n)(x)d(x) \leqq \int_{-u/2}^{u/2} h_n(x)d(x)$$

where $|E| \leqq u$.

Therefore,

$$\int_0^u (f_1 * \cdots * f_n)^*(x)d(x) \leqq \int_0^u h_n^*(x)d(x) \quad \text{for} \quad u \geqq 0 .$$

By Lemma 2.2 this implies

$$\int_0^u \psi((f_1 * \cdots * f_n)^*(x))d(x) \leqq \int_0^u \psi(h_n^*(x))d(x) \quad \text{for} \quad u \geqq 0 .$$

Hence,

$$\int_0^\infty \psi[(f_1 * \cdots * f_n)^*(x)]d(x) \leqq \int_{-\infty}^{+\infty} \psi(h_n(x))d(x) .$$

Thus by (1.4),

$$\int_{-\infty}^{+\infty} \psi[(f_1 * \cdots * f_n)(x)]d(x) \leqq \int_{-\infty}^{+\infty} \psi(h_n(x))d(x) .$$

To say that $g \in L_p(0, \infty)$, means

$$\| g \|_p = \left\{ \int_0^\infty | g(x) |^p d(x) \right\}^{1/p} < \infty .$$

We shall henceforth use the symbol $\| \cdot \|_p$ to mean the $p$th norm over $L_p(0, \infty)$.

LEMMA 2.4. *If $g \in L_p(0, \infty)$ for $p > 1$, then*

$$\frac{1}{p+1} \| g \|_p \leqq \left\| \frac{1}{x} \int_0^x g - g(x) \right\|_p \leqq (p' + 1) \| g \|_p$$

*where $1/p + 1/p' = 1$. In particular, for the case $p = 2$ we get that*

$$\| g \|_2 = \left\| \frac{1}{x} \int_0^x g - g(x) \right\|_2 .$$

*Proof.* Let $g_n(x) = \begin{cases} g(x) & g(x) \leqq n, |x| \leqq n \\ 0 & \text{elsewhere} \end{cases}$

and suppose that $f \in L_1 \cap L_\infty$. We see that

$$\int_0^u f(t)g_n(t)dt = \frac{1}{u}\int_0^u f\int_0^u g_n + \int_0^u \frac{d(x)}{x^2}\Big(\int_0^x f - xf\Big)\Big(\int_0^x g_n - xg_n\Big)$$

by simply differentiating both sides for $u \in [\eta, m]$ and then letting $\eta \to 0$. Therefore,

$$\int_0^\infty f(t)g_n(t)dt = \int_0^\infty \frac{d(x)}{x^2}\Big(\int_0^x f - xf\Big)\Big(\int_0^x g_n - xg_n\Big) . \tag{2.5}$$

Since $1/p + 1/p' = 1$ we have,

$$\begin{aligned} \| g_n \|_p &= \sup_{\substack{\|f\|_{p'} \leqq 1 \\ f \in L_1 \cap L_\infty}} \int_0^\infty f(t)g_n(t)dt \leqq \sup_{\|f\|_{p'} \leqq 1} \Big\|\frac{1}{x}\int_0^x f - f(x)\Big\|_{p'} \cdot \Big\|\frac{1}{x}\int_0^x g_n - g_n\Big\|_p \\ &\leqq (p+1)\Big\|\frac{1}{x}\int_0^x g_n - g_n\Big\|_p . \end{aligned}$$

We also have that

$$\begin{aligned} \Big\|\frac{1}{x}\int_0^x g_n - g_n(x)\Big\|_p &\leqq \Big\|\frac{1}{x}\int_0^x g_n\Big\|_p + \| g_n \|_p \\ &\leqq (p'+1) \| g_n \|_p . \end{aligned}$$

Therefore,

$$\frac{1}{(p+1)} \| g_n \|_p \leqq \Big\|\frac{1}{x}\int_0^x g_n - g_n(x)\Big\|_p \leqq (p'+1) \| g_n \|_p \quad \text{for each} \quad n .$$

Since $g \in L_p(0, \infty)$, we finally see that

$$\frac{1}{(p+1)} \| g \|_p \leqq \Big\|\frac{1}{x}\int_0^x g - g(x)\Big\|_p \leqq (p'+1) \| g\|_p \ (p>1) .$$

The case $p = 2$ follows immediately from (2.5).

DEFINITION 2.6. We say that a function $\psi \in \bigvee (p, q)$ on $[a, b]$ if there exist $1 < p$ and $1 < q$ such that $\psi^{1/p} \in \bigvee$ on $[a, b]$ and $\psi^{1/q} \in \bigwedge$ on $[a, b]$.

For example, the functions $\psi(x) = x^r$ with $r > 1$ belong to $\bigvee(p, q)$ on $[0, \infty]$ where $p = r - \varepsilon$ and $q = r + \varepsilon$ for some suitably chosen $\varepsilon > 0$.

LEMMA 2.7. *If* $\psi \in \bigvee(p, q)$ *on* $[0, \infty)$ *and* $\int_0^\infty \psi(f^*(t))dt < \infty$, *then*

$$\frac{1}{(q+1)^q}\int_0^\infty \psi(f^*(x))d(x) \leqq \int_0^\infty \psi\Big(\frac{1}{x}\int_0^x f^* - f^*(x)\Big)d(x)$$
$$\leqq (p'+1)^p \int_0^\infty \psi(f^*(x))d(x)\ ,$$

*where* $1/p + 1/p' = 1$.

*Proof*. First, since $\psi \in \bigvee(p, q)$ on $[0, \infty)$, this implies there exist $p, q > 1$ such that $\psi^{1/p} \in \bigvee$ and $\psi^{1/q} \in \bigwedge$. Thus,

$$\psi^{1/p}\Big(1/x\int_0^x f^* - f^*(x)\Big) \leqq \psi^{1/p}\Big(1/x\int_0^x f^*\Big) - \psi^{1/p}(f^*(x))\ .$$

Now by applying Jensen's inequality we see

$$\psi^{1/p}\Big(1/x\int_0^x f^* - f^*(x)\Big) \leqq \frac{1}{x}\int_0^x \psi^{1/p}(f^*) - \psi^{1/p}(f^*(x))\ . \tag{2.8}$$

Now by (2.8) we get that

$$\Big\|\psi^{1/p}\Big(1/x\int_0^x f^* - f^*(x)\Big)\Big\|_p \leqq \Big\|\frac{1}{x}\int_0^x \psi^{1/p}(f^*) - \psi^{1/p}(f^*(x)\Big\|_p$$

and by applying Lemma 2.4 we find

$$\leqq (p'+1)\,\|\,\psi^{1/p}(f^*)\,\|_p\ .$$

Since $\psi^{1/q} \in \bigwedge$ we have that

$$\psi^{1/q}\Big(\frac{1}{x}\int_0^x f^* - f^*(x)\Big) \geqq \psi^{1/q}\Big(\frac{1}{x}\int_0^x f^*\Big) - \psi^{1/q}(f^*(x))$$
$$\geqq \frac{1}{x}\int_0^x \psi^{1/q}(f^*) - \psi^{1/q}(f^*(x))\ .$$

Therefore by Lemma 2.4 we have that

$$\Big\|\psi^{1/q}\Big(\frac{1}{x}\int_0^x f^* - f^*(x)\Big)\Big\|_q \geqq \Big\|\frac{1}{x}\int_0^x \psi^{1/q}(f^*) - \psi^{1/q}(f^*(x))\Big\|_q$$
$$\geqq \Big(\frac{1}{q+1}\Big)\,\|\,\psi^{1/q}(f^*)\,\|_q\ .$$

THEOREM 2.9. *If* $\psi \in \bigvee(p, q)$ *on* $[0, \infty)$ *and* $\int_0^\infty \psi(R_n(x))d(x) < \infty$, then

$$\frac{1}{(p'+1)}\cdot\frac{1}{(2^{n-1})q}\int_0^\infty \psi[x^{n-1}(g_1^{**} - g_1^*)\cdots(g_n^{**} - g_n^*)]d(x)$$
$$\leqq \sup_{f_i \sim g_i^*}\int_{-\infty}^{+\infty} \psi((f_1 * \cdots * f_n))(x)d(x)$$
$$\leqq (q+1)^q \int_0^\infty \psi[x^{n-1}(g_1^{**} - g_1^*)\cdots(g_n^{**} - g_n^*)]d(x)$$

*where* $1/p' + 1/p = 1$.

*Proof.* By (1.8), where $1/2^{n-1} \leqq K \leqq 1$, we see that

$$\int_0^u h_n^*(x)d(x) \leqq \int_0^u R_n(x)d(x) \leqq 2^{n-1}\int_0^u h_n^*(x)d(x) \quad \text{for} \quad u \geqq 0 .$$

Since both $R_n$ and $h_n^*$ are decreasing, this implies by Lemma 2.2

$$\begin{aligned}\int_0^\infty \psi(h_n^*)(x)d(x) &\leqq \int_0^\infty \psi(R_n)(x)d(x) \\ &\leqq \int_0^\infty \psi(2^{n-1}h_n^*)(x)d(x) .\end{aligned} \tag{2.10}$$

Now applying Lemma 2.7 to (2.10) we see that there exist a $p > 1$ and $q > 1$ such that

$$\frac{1}{(q+1)^q}\int_0^\infty \psi(h_n^*)(x)d(x) \leqq \int_0^\infty \psi\Big(\frac{1}{x}\int_0^x R_n - R_n(x)\Big) \tag{2.11}$$

and

$$\int_0^\infty \psi\Big(\frac{1}{x}\int_0^\infty R_n - R_n(x)\Big) \leqq (p'+1)^p\int_0^\infty \psi(2^{n-1}h_n^*(x))d(x) . \tag{2.12}$$

Since $\psi^{1/q} \in \Lambda$, this implies

$$\psi^{1/q}(t) = \int_0^t \varphi(v)dv$$

where $0 < \varphi$ and is decreasing on $[0, \infty)$.

Therefore for $a \geqq 1$,

$$\psi(at) \leqq a^q\psi(t) . \tag{2.13}$$

Now applying (2.13) to the right side of (2.12) we conclude

$$\int_0^\infty \psi\Big(\frac{1}{x}\int_0^x R_n - R_n(x)\Big) \leqq (p'+1)^p(2^{n-1})^q\int_0^\infty \psi(h_n^*) . \tag{2.14}$$

We then apply Lemma 1.9(d) and Lemma 2.3 to (2.11) and (2.14) to obtain our result.

COROLLARY 2.15. *If*

$$\int_0^\infty (R_n(x))^r d(x) < \infty ,$$

*then*

$$\frac{1}{(r'+1)}\frac{1}{(2^{n-1})^r}\int_0^{\infty}[x^{n-1}(g_1^{**}-g_1^*)\cdots(g_n^{**}-g_n^*)]^r d(x)$$
$$\leqq \sup_{f_i\sim g_i^*}\int_{-\infty}^{+\infty}[(f_1*f_2*\cdots*f_n)(x)]^r d(x)$$
$$\leqq (r+1)^r\int_0^{\infty}[x^{n-1}(g_1^{**}-g_1^*)\cdots(g_n^{**}-g_n^*)]^r d(x)$$

*where* $1/r+1/r'=1$ *and* $r\geqq 1$.

*Proof*. Apply Theorem 2.9 with $\psi(u)=u^r$ and $q=r+\varepsilon$, $p=r-\varepsilon$ with $\varepsilon>0$. Then, let $\varepsilon$ go to zero.

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